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WEAK COMPACTNESS IN $L^1(\lambda)$ AND INJECTIVE BANACH SPACES

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ABSTRACT

The isomorphic embedding of the Banach space $l^1(\Gamma)$ into injective Banach spaces is investigated.

Introduction

In the present paper we study the isomorphic structure of injective Banach spaces and in particular we prove some results concerning the following Rosenthal's problem.

PROBLEM. Let X be an injective Banach space with $\dim X = \alpha$.

(a) Is $\alpha^\omega = \alpha$?

(b) Is $l^1\alpha$ isomorphic to a subspace of X?

(c) Is X^* isomorphic to $(\Sigma_{2^\alpha} \bigoplus L^1\{0,1\}^\alpha)_1$?

As we have proved in [2] questions (b) and (c) are equivalent and a consequence of the results of this paper is that (b) implies (a). So what we need in the above problem is an answer about the possibility of embedding of $l^1\alpha$ into X where $\dim X = \alpha$. In this direction we prove in §2 (Theorem 2.8) that if an injective Banach space X contains $l^1\alpha_n$ for a sequence $\{\alpha_n\}_{n=1}^\infty = 1$ of cardinals, then X contains also $l^1\alpha^\omega$ where $\alpha = \sup\{\alpha_n\}$. A consequence of this result is an affirmative answer in Rosenthal's problem for all injective subspaces of $L^\infty(\mu)$ for μ finite measure.

A basic tool for the proof of this theorem is a result (Lemma 2.1) about the structure of weakly compact subsets of $L^1(\lambda)$ for some measure λ.

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§1. Preliminaries

Let X be a Banach space and Y be a subspace of X; we say that Y is a complemented subspace of X if there is a bounded projection P from X onto Y, i.e. P is a bounded linear operator with $P(y) = y$ for all $y \in Y$.

A Banach space Y is injective if whenever Y is a subspace of a Banach space X there is a bounded projection $P : X \to Y$.

Let $T : X \to Y$ be a linear bounded operator between the Banach spaces X and Y. We denote by $T^* : Y^* \to X^*$ the conjugate operator of T. A Banach space Y is isomorphic to a subspace of a Banach space X if there is a bounded linear operator $T : Y \to X$ one-to-one with closed range.

Let Ω be a compact space. By $C(\Omega)$ we denote the Banach space of continuous real functions on Ω and $M(\Omega)$ the Banach space of regular Borel measures on Ω. Via Riesz represent theorem we identify the $M(\Omega)$ with $C(\Omega)^*$.

For a set I we denote by μ_I the product measure which is defined on the product space $\{0,1\}^I$ from the family $\{\mu_i : i \in I\}$ where $\mu_i(\{0\}) = \mu_i(\{1\}) = \frac{1}{2}$ for all $i \in I$.

For each $1 \leqq p \leqq \infty$ we denote by $L^p\{0,1\}^I$ the space $L^p(\mu_I)$. Let $I \neq \emptyset$ be a set; the *Walk functions* Π_M on $C(\{0,1\}^I)$ are defined for each finite subset M of I in the following way: for the empty set we set $\Pi_\emptyset(x) = 1$ for all $x \in \{0,1\}^I$; for a $i \in I$ we set $\Pi_{\{i\}}(x) = 1$ if $x(i) = 1$ and $\Pi_{\{i\}}(x) = -1$ otherwise. If M is a finite subset of I we set $\Pi_M = \Pi_{i \in M} \Pi_{\{i\}}$. Let $1 \leqq p \leqq \infty$ and Λ be a subset of I; we denote by $E^p_\Lambda : L^p\{0,1\}^I \to L^p\{0,1\}^\Lambda$ the conditional expectation projection. The operator E_Λ is alternatively described as follows: if $\Lambda = \emptyset$ then

$$E_\Lambda(f) = \left(\int f d\mu\right) \Pi\emptyset;$$

if $\Lambda \neq \emptyset$ for q such that $1/p + 1/q = 1$ we set $I_\Lambda : L^q\{0,1\}^\Lambda \to L^q\{0,1\}^I$ to be the usual embedding, then

$$E_\Lambda = I^*_\Lambda \big|_{L^p\{0,1\}^I}.$$

Given a set Γ, $c_0(\Gamma)$ denotes the Banach space of all real valued functions f defined on Γ such that for $\varepsilon \geqq 0$ there is a finite subset F of Γ with $|f(\gamma)| < \varepsilon$ for all $\gamma \in \Gamma \setminus F$ with the supremum norm; $l^1(\Gamma)$ denotes all elements of $c_0(\Gamma)$ for which $\Sigma_{\gamma \in \Gamma} |f(\gamma)| < \infty$ with $\|f\| = \Sigma_{\gamma \in \Gamma} |f(\gamma)|$ and $l^\infty(\Gamma)$ is the space of all bounded real valued functions.

By the canonical or usual basis of $l^1(\Gamma)$ (resp. $c_0(\Gamma)$) we refer to $\{e_\gamma : \gamma \in \Gamma\}$ where $e_\gamma(\delta) = 1$ if $\gamma = \delta$ and $e_\gamma(\delta) = 0$ if $\gamma \neq \delta$. A subset $\{b_\gamma : \gamma \in \Gamma\}$ of a Banach space X is said to be equivalent to the canonical basis of $l^1(\Gamma)$ if the correspondence $T:\{e_\gamma:\gamma\in\Gamma\}\to\{b_\gamma:\gamma\in\Gamma\}$ defined by $Te_\gamma=b_\gamma$ for $\gamma\in\Gamma$ can be extended to an isomorphism of $l^1(\Gamma)$ with the closed linear span of $\{b_\gamma:\gamma\in\Gamma\}$.

Let $\{b_\gamma:\gamma\in\Gamma\}$ be a uniformly bounded family in a Banach space X; then it is equivalent to the usual basis of $l^1(\Gamma)$ iff there is a $\theta>0$ such that for every finite choice $\gamma_1,\cdots,\gamma_\kappa$ of index pairwise difference and $r_1,r_2,\cdots,r_\kappa$ real numbers it follows that

$$\left\|\sum_{i=1}^{\kappa} r_i b_{\gamma i}\right\| \geqq \theta\sum_{i=1}^{\kappa}|r_i|.$$

Let $\{X_i:|\, i\in I\}$ be a family of Banach spaces and $p=0,1,\infty$ then we denote by $(\Sigma_{i\in I}\oplus X_i)_p$ the Banach space of all functions $x=(x_i:i\in I)$ with $x_i\in X_i$ and the real valued function $(\|x_i\|:i\in I)$ belongs respectively to $l^p(I)$ if $p=1$, or ∞ and $c_0(I)$ if $p=0$.

A compact space Ω is called extremally disconnected if for every open U subset of Ω the set $\bar U$ (the closure of U) is also open. For a compact space Ω we set $S(\Omega)$ to be the smaller cardinal κ such that every family of non-empty open pairwise disjoint open subsets of Ω has cardinality less than κ. Also, for a Banach space X we set $\Sigma(X)$ to be the smaller cardinal κ such that every weakly compact subset of X has cardinality less than κ. It follows from a result of Rosenthal in [9] that for every compact space Ω, $\Sigma(C(\Omega))=S(\Omega)$.

Given an infinite cardinal α, its cofinality, denoted $\mathrm{cf}(\alpha)$, is the least cardinal β such that α is the cardinal sum of β many cardinals each smaller than α. A cardinal α is regular if $\alpha=\mathrm{cf}(\alpha)$. The least cardinal strictly greater than β is denoted by β^+. A cardinal α is successor cardinal if it is of the form $\alpha=\beta^+$ for some cardinal β; the cardinalty of the natural numbers is denoted by ω. The cardinality of a set A is denoted by $|A|$.

We denote by $\mathcal{P}(\alpha)$ (resp. $\mathcal{P}_\kappa(\alpha)$) the set of subsets of α (resp. the set of subsets of α of cardinaltiy less than κ). The cardinality of $\mathcal{P}(\alpha)$ is denoted by 2^α and the cardinality of $\mathcal{P}_{\kappa^+}(\alpha)$ is denoted by α^κ. If α, κ are cardinals, then α is called strongly κ inaccessible if $\beta^\lambda<\alpha$ for any $\beta<\alpha$ and $\lambda<\kappa$. If in addition $\alpha>\kappa$ then we write $\alpha\gg\kappa$ (or $\kappa\ll\alpha$).

For set-theoretic background we refer the reader to [3].

We make use, in the following, of the next two infinite combinatoric results of Erdös–Rado and Hajnal, respectively.

1.1. Theorem. [4]. *Let α, κ be cardinals such that α is regular and $\alpha\gg\kappa$ and $f:\alpha\to\mathcal{P}_\kappa(\alpha)$ be a function. Then there is a set A subset of α with $|A|=\alpha$ and $N\subset\alpha$ such that for every $\xi_1\neq\xi_2\in A$, $f(\xi_1)\cap f(\xi_2)=N$.*

1.2. THEOREM. [5]. *Let α, κ be cardinals with $\alpha > \kappa$ and $f : \alpha \to \mathcal{P}_\kappa(\alpha)$ be a function. Then there is a set A subset of α with $|A| = \alpha$ and if $\xi_1 \neq \xi_2 \in A$, $\xi_1 \notin f(\xi_2)$.*

§2. This section is devoted to the proof of the main result of this paper. We start with the following lemma about the structure of the weakly compact subset of $L^1(\lambda)$ spaces.

2.1. LEMMA. *Let $(S, \mathcal{Z}, \mu)$ be a space of probability measure and α be an infinite cardinal. Let, also, K be a relatively weakly compact subset of $L^1(\lambda)$, with $|K| = \alpha$, $0 < \|x\| < \theta$ for all $x \in K$. Then there is a measure λ in $L^1(\lambda)$ such that $\|\lambda\| \geqq \theta$ and for each $A \in \mathcal{Z}$ with $\lambda(A) > \varepsilon > 0$ there is a $\Lambda \subset K$ with $|\Lambda| = \alpha$ such that $|x|(A) > \varepsilon$ for all $x \in \Lambda$.*

PROOF. Since the set K is relatively weakly compact so is the set $L = \{|x| : x \in K\}$. (Indeed, this happens iff K is uniformly absolutely continuous, see Dunford Schwartz.) Now we claim that there is a measure λ in $L^1(\lambda)$ such that $|L \cap U| = |L|$ for every weak neighborhood U of λ; otherwise we can find a finite set $\{\lambda_i\}_{i=1}^n$ of elements of $\bar{L}$ and $\{U_i\}_{i=1}^n$ such that each U_i is weak neighborhood of λ_i, $|U_i \cap L| < |L|$ and $L \subset \bigcup_{i=1}^n U_i$, a contradiction. So the claim is correct and we easily verify that the measure λ has the desired properties. The proof is now complete.

2.2. LEMMA. *Let α be a regular ω^+ inaccessible cardinal and $\{\mu_\gamma : \gamma \in \Gamma\}$ be a family of finite positive measure. Assume that*

$$T : \Big(\sum_{\xi<\alpha} \oplus L^1_\xi\{0,1\}^\alpha\Big)_1 \to \Big(\sum_{\gamma\in\Gamma} \oplus L^1(\mu_\gamma)\Big)_1$$

is an isomorphic embedding with $\|T^{-1}\|^{-1} = \theta$ and $\|T\| = 1$. Then there is a $\Lambda \subset \alpha$ with $|\Lambda| = \alpha$ and a family $\{\Delta_\xi : \xi \in \Lambda\}$ of pairwise disjoint subsets of Γ such that if $\Delta = \bigcup_{\xi\in\Lambda} \Delta_\xi$ then

$$P_\Delta \circ T : \Big(\sum_{\xi\in\Lambda} \oplus L^1\{0,1\}^\alpha\Big)_1 \to \Big(\sum_{\gamma\in\Delta} \oplus L^1(\mu_\gamma)\Big)_1$$

is an isomorphic embedding with $\|(P_\Delta \circ T)^{-1}\|^{-1} \geqq \theta/2$ and for $\xi_1 \neq \xi_2$ elements of Λ we have that

$$P_{\Delta\xi_1} \circ T(x) = 0 \qquad \textit{for all } x \in L^1_{\xi_2}\{0,1\}^\alpha.$$

PROOF. For each $\xi < \alpha$ we set M_ξ be the countable subset of Γ such that $T(L^1_\xi\{-1,1\}^\alpha) \hookrightarrow (\Sigma_{\gamma\in M_\xi} \oplus L^1(\mu_\gamma))_1$. For the family $\{M_\xi : \xi < \alpha\}$ we apply the

Erdös–Rado Δ systems lemma and we choose $\Lambda_1 \subset \alpha$ with $|\Lambda_1| = \alpha$ and $M \subset \Gamma$ such that $M_{\xi_1} \cap M_{\xi_2} = M$ for $\xi_1 \neq \xi_2$ in Λ_1. If $M = \varnothing$ then setting $\Lambda_1 = \Lambda$ and $M_\xi = \Delta_\xi$ we have the desired result.

Assume now that $M \neq \varnothing$ and for each $\xi \in \Lambda_1$ we set $\Delta_\xi = M_\xi - M$. We claim that there is a most countable set Λ_α subset of Λ_1 such that if $\xi \in \Lambda_1 \setminus \Lambda_2$ then for each $x \in L^1_\xi\{-1,1\}^\alpha$ with $\|x\| = 1$, $\|P_{\Delta_\xi} \circ T(x)\| \geqq \theta/2$ holds.

In fact, assuming the contrary we choose an uncountable family $\{\xi_\sigma : \sigma < \omega^+\}$ of elements of Λ_1 and for each $\sigma < \omega^+$, $x_{\xi_\sigma} \in L^1_{\xi_\sigma}\{-1,1\}^\alpha$ such that $\|x_{\xi_\sigma}\| = 1$ and

$$\|T(x_{\xi_\sigma}) - P_M \circ T(x_{\xi_\sigma})\| < \frac{\theta}{2}.$$

But, since the family $\{T(x_{\xi_\sigma}) : \sigma < \omega^+\}$ is equivalent to the usual basis of $l^1\omega^+$ with constants 1 and θ, the same is true for the family $\{P_M \circ T(x_{\xi_\sigma}) : \sigma < \omega^+\}$ with constants 1 and $\theta/2$. So $l^1(\omega^+)$ is contained isomorphically in $(\Sigma_{\gamma\in M} \oplus L^1(\mu_\gamma))_1$, a contradiction, since M is countable. So the claim is correct and setting $\Lambda = \Lambda_1 \setminus \Lambda_2$ we easily verify that this set satisfies the conclusion.

2.3. Lemma. *Let $\{\mu_\gamma : \gamma \in \Gamma\}$ be a family of finite measures and $\{\alpha_n : n < \omega\}$ be a strictly increasing sequence of regular ω^+ inaccessible cardinals such that the following holds: there is $\theta > 0$, a real number, such that for every $n < \omega$ there is*

$$T_n : \left(\sum_{\xi<\alpha_n} \oplus L^1\{0,1\}^{\alpha_n}\right)_1 \to \left(\sum_{\gamma\in\Gamma} \oplus L^1(\mu_\gamma)\right)_1$$

isomorphic embedding with $\|T_n\| = 1$ and $\|T_n^{-1}\|^{-1} \geqq \theta$. Then there is a sequence $\{\xi_1, \xi_2, \cdots, \xi_n, \cdots : n < \omega\}$ and a sequence $\{\Delta_1, \Delta_2, \cdots, \Delta_n : n < \omega\}$ of pairwise disjoint subsets of Γ such that

(i) $\xi_n < \alpha_n$,

(ii) $P_{\Delta_n} \circ T_n : L^1_{\xi_n}\{0,1\}^{\alpha_n} \to (\Sigma_{\gamma\in\Delta_n} \oplus L^1(\mu_\gamma))_1$ *is an isomorphic embedding with* $\|(P_{\Delta_n} \circ T_n)^{-1}\|^{-1} \geqq \theta/2$,

(iii) *if $n_1 \neq n_2$ then $P_{\Delta_{n_1}} \circ T_{n_1}(x) = 0$ for all $x \in T_{n_2}(L^1_{\xi_{n_2}}\{0,1\}^{\alpha_{n_2}})$.*

Proof. For every $n < \omega$, using Lemma 2.2 we choose $\Lambda_n \subset \alpha_n$ and a family $\{\Delta^n_\xi : \xi \in \Lambda_n\}$ such that $\Delta^n_{\xi_1} \cap \Delta^n_{\xi_2} = \varnothing$ for all $\xi_1 \neq \xi_2$ in Λ_n and for $\Delta_n = \bigcup_{\xi\in\Lambda_n} \Delta^n_\xi$ the map

$$P_\Delta \circ T_n : \left(\sum_{\xi\in\Lambda_n} \oplus L^1_\xi\{0,1\}^{\alpha_n}\right)_1 \to \left(\sum_{\gamma\in\Delta_n} \oplus L^1(\mu_\gamma)\right)_1$$

is an isomorphic embedding with $\|(P_\Delta \circ T_n)^{-1}\|^{-1} \geqq \theta/2$ and

$$P_\Delta \circ T_n(L^1_\xi\{0,1\}^{\alpha_n}) \hookrightarrow \left(\sum_{\gamma\in\Delta^n_\xi} \oplus L^1(\mu_\gamma)\right).$$

Inductively we choose $\{\xi_n : n < \omega\}$ such that $\xi_n \in \Lambda_n$ and if $n_1 < n_2 < \omega$ then $\Delta_{\xi_{n_1}} \cap M_{\xi_{n_2}} = \varnothing$, $M_{\xi_{n_1}} \cap \Delta_{\xi_{n_2}} = \varnothing$ where $M_{\xi_{n_1}}$ denotes the countable subset of Γ on which the space $T_n(L^1_{\xi_n}\{0,1\}^{\alpha_n})$ depends.

It is easy to verify that the sequences $\{\xi_n : n < \omega\}$, $\{\Delta^n_{\xi_n} : n < \omega\}$ satisfy the conclusion and the proof is complete.

2.4. Remark. If for the previous sequence $\{\xi_n : n < \omega\}$ we define an operator

$$T : \Big(\sum_{n<\omega} \oplus L^1_{\xi_n}\{0,1\}^{\alpha_n}\Big)_1 \to \Big(\sum_{\gamma\in\Gamma} \oplus L^1(\mu_\gamma)\Big)_1$$

by the rule $T(x_n : n < \omega) = \Sigma_{n<\omega} T_n(x_n)$, then it is obvious that T is an isomorphism with $\|T\| = 1$ and $\|T^{-1}\|^{-1} \geqq \theta/2$.

2.5. Lemma. *Let S be an extremally disconnected compact space and $\{U_1, U_2, \cdots, U_n, \cdots : n < \omega\}$ be a sequence of pairwise disjoint clopen sets of S and $\{\lambda_1, \lambda_2, \cdots \lambda_n, \cdots : n < \omega\}$ a sequence of positive measure in $M(S)$. Then there is a subsequence $\{U_{n_1}, U_{n_2}, \cdots, U_{n_\kappa}, \cdots : \kappa < \omega\}$ such that $\lambda_n(\overline{\bigcup_{\kappa=1}^{\omega} U_{n_k}} \setminus \bigcup_{k=1}^{\omega} U_{n_\kappa}) = 0$ for all $n < \omega$.*

Proof. For $A \subseteq \omega$ we denote $W_A = \overline{\bigcup_{n\in A} U_n} \setminus \bigcup_{n\in A} U_{n_A}$. Since S is extremally disconnected $A \cap B = \varnothing$ implies that $W_A \cap W_B = \varnothing$ and obviously the same holds if $A \cap B$ is finite. Let $\{A_\xi : \xi < \omega^+\}$ be an almost disjoint family of infinite subsets of N. Then for each n there are at most countably many ξ's such that $\lambda_n(W_{A_\xi}) \neq 0$. Hence there is ξ such that $\lambda_n(W_{A_\xi}) = 0$ for all n.

The following result is contained in [2] (theorem 5.1).

2.6. Theorem. *Let α be an ω^+ inaccessible cardinal with $\mathrm{cf}(\alpha) > \omega$ and $\{x_\xi : \xi < \alpha\}$ be a family of norm one elements of $L^\infty\{0,1\}^\alpha$ with $\|x_{\xi_1} - x_{\xi_2}\| > \theta > 0$. Then there is an $A \subset \alpha$ with $|A| = \alpha$ and a family $\{y_n : n \in A\}$ of elements of $L^\infty\{0,1\}^\alpha$ such that*

(i) *for each $n \in A$ there are $\xi_{(n,1)}$, $\xi_{(n,2)}$ such that $y_n = x_{(\xi,1)} - x_{(\xi,2)}$.*

(ii) *if $c_1, \cdots, c_\kappa$ are real numbers and $n_1 \neq \cdots \neq n_\kappa$ are elements of A, then*

$$2\sum_{i=1}^{\kappa} |c_i| \geqq \Big\|\sum_{i=1}^{\kappa} c_i y_{n_i}\Big\| \geqq \frac{\theta}{172}\sum_{i=1}^{\kappa} |c_i|.$$

2.7. Lemma. *Let S be a compact extremally disconnected space and X a complemented subspace of $C(S)$. Let, also, $\{\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots : n < \omega\}$ be an increasing sequence of regular ω^+ inaccessible cardinals and $\alpha = \sup\{\alpha_n : n < \omega\}$. We denote by $P : C(S) \to X$ a projection onto X and assume that there is $\{U_1, U_2, \cdots, U_n, \cdots : n < \omega\}$, a sequence of clopen mutually disjoint subsets* *of* S, $T: (\Sigma_{n=1}^{\omega} \oplus L^1\{-1,1\}^{\alpha_n})_1 \to X^*$ *isomorphic embedding* $\delta > 0$ *and* $\{K_1, K_2, \cdots, K_n, \cdots : n < \omega\}$ *such that*

(i) K_n *is a weakly discrete subset of the unit ball of* $L^1\{0,1\}^{\alpha_n}$ *with* $|K_n| = \alpha_n$ *and* $K_n \cup \{0\}$ *is weakly compact,*

(ii) *for each* $n < \omega$ *and* $x \in K_n$

$$|P^* \circ T(x)|(U_n) > \delta.$$

Then X *has a subspace isomorphic to* $l^1\alpha^{\omega}$.

PROOF. For each $n < \omega$ we choose a measure $\lambda_n \in M_1^+(S)$ such that $P^* \circ T(L^1\{-1,1\}^{\alpha_n})$ is contained in $L^1(\lambda_n)$ and using Lemma 2.5 we fined $\{U_{n_1}, U_{n_2}, \cdots, U_{n_\kappa}, \cdots : \kappa < \omega\}$ such that $\lambda_n(\overline{\bigcup_{k=1}^{\omega} U_{n_\kappa}} \setminus \bigcup_{k=1}^{\omega} U_{n_\kappa}) = 0$ for all $n < \omega$. For simplicity we assume that this happens for the sequence $\{U_1, U_2, \cdots, U_n, \cdots : n < \omega\}$.

Let $n < \omega$ fix and $\{x_\xi^n : \xi < \alpha_n\}$ be a well-order onto the set K_n. Since for each $\xi < \alpha_n$, $|P^* \circ T(x_\xi^n)|(U_n) > \delta$, there is an f_ξ^n such that

(i) $\|f_\xi^n\| \leqq 1$,

(ii) support $(f_\xi^n) \subset U_n$,

(iii) $\int f_\xi^n d(P^* \circ Tx_\xi^n) > \delta$.

Also, since the family $\{x_\xi^n : \xi < \alpha_n\}$ is weakly discrete and has one point compactification, the element $0 \in L^1\{-1,1\}^{\alpha_n}$ we have that for each $\xi < \alpha_n$ the set $\{f_\zeta^n : \int f_\zeta^n d(P^* \circ Tx_\xi^n) \neq 0\}$ is at most countable. So using Hajnal's theorem we pass to a $\Lambda_n' \subset \Lambda_n$ with $|\Lambda_n'| = \alpha_n$ and for $\xi_1 \neq \xi_2 \in \Lambda_n$,

$$\int f_{\xi_1}^n d(P^* \circ Tx_{\xi_2}^n) = 0$$

holds. Now, let $T^*: X \to (\Sigma_{n=1}^{\omega} \oplus L^\infty\{-1,1\}^{\alpha_n})_\infty$ be the restriction of the conjugate of T onto the space X. Also, in the rest, we denote by

$$P_n : \left(\sum_{n=1}^{\omega} \oplus L^\infty\{-1,1\}^{\alpha_n}\right)_\infty \to L^\infty\{-1,1\}^{\alpha_n}$$

the usual projection.

Using finite induction we choose a sequence $\{L_1, L_2, \cdots, L_n, \cdots : n < \omega\}$ such that $|L_n| = \alpha_n$ and with the following properties:

(i) $L_n = \{g_\sigma^n : \sigma < \alpha_n\}$ where

$$g_\sigma^n = f_{\xi(\sigma,1)}^n - f_{\xi(\sigma,2)}^n$$

and for $\sigma_1 \neq \sigma_2 \neq \cdots \neq \sigma_\kappa$ finite choice of index and $c_1, \cdots, c_\kappa$ real numbers

$$2\|P\| \sum_{i=1}^{\kappa} |c_i| \leqq \left\| \sum c_i g_{\sigma_i}^n \right\| \leqq \frac{\delta}{172} \sum_{i=1}^{\kappa} |c_i|.$$

(ii) For $n > 1$ setting

$$I_n = \{\xi < \alpha_n : \exists n_1 < n \text{ and } g_\sigma^{n_1} \in L_{n_1} \text{ such that } P_n \circ T^* \circ P(g_\sigma^{n_1}) \text{ depends on the coordinate } \xi\}$$

it follows that

$$E_{I_n}^\infty \circ P_n \circ T^* \circ P(g_\sigma^n) = 0 \qquad \text{for all } g_\sigma^n \in L_n.$$

(iii) For $n > 1$ and $n_1 < n$ we have that

$$P_{n_1} \circ T^* \circ P(g_\sigma^n) = 0 \qquad \text{for all } g_\sigma^n \in l_n.$$

Let $n > 1$ and assume that for all $\kappa < n$, L_κ has been constructed. (The case $n = 1$ is similar and more simple.) We consider the family

$$\{P_n \circ T^* \circ P(f_\xi^n) : \xi \in \Lambda_n\}$$

and we observe that if $\xi_1 \neq \xi_2$, then

$$(*) \qquad \begin{aligned} &\|P_n \circ T^* \circ P(f_{\xi_1}^n - f_{\xi_2}^n)\| \geqq \\ &\geqq \left| \int f_{\xi_1} d(P^* \circ T(x_{\xi_1}^n)) - \int f_{\xi_2} d(P^* \circ T(x_{\xi_1}^n)) \right| > \delta. \end{aligned}$$

Also, since $\alpha_n > \Sigma_{k<n} \alpha_\kappa$ and α_n is regular ω^+ inaccessible cardinal there is a $\Lambda_n' \subset \Lambda_n$ such that $|\Lambda_n'| = |\Lambda_n|$ and for $\xi_1, \xi_2 \in \Lambda_n'$, $\kappa < n$ the following hold:

(a) $P_\kappa \circ T^* \circ P(f_{\xi_1}^n) = P_\kappa \circ T^* \circ P(f_{\xi_2}^n)$,

(b) $E_{I_n}^\infty \circ P_n \circ T^* \circ P(f_{\xi_1}^n) = E_{\xi_n}^\infty \circ P_n \circ T^* \circ P(f_{\xi_2}^n)$.

For the family $\{P_n \circ T^* \circ P(f_\xi^n) : \xi \in \Lambda_n'\}$ we apply Theorem 2.5 and we choose a set L_n such that $|L_n| = \alpha_n$ and L_n satisfies the inductive assumption (i). From (a) and (b) it follows that L_n satisfies, also, assumptions (ii) and (iii) and so the inductive construction is complete.

Let $\{A_\xi : \xi < \alpha^\omega\}$ be a subset of the set $\Pi_{n<\omega} L_n$ such that for $\xi_1 < \xi_2 < \alpha^\omega$, $|\{n < \omega : A_{\xi_1}(n) = A_{\xi_2}(n)\}| < \omega$.

Each A_ξ has the form

$$\{g_1^\xi, \cdots, g_n^\xi, \cdots : n < \omega\}$$

where g_n^ξ is supported by the clopen set U_n. We set g_ξ to be the usual extension of the above sequence to an element of $C(S)$ such that $g_\xi(s) = 0$ for all $s \in S \setminus \overline{\bigcup_{n<\omega} U_n}$. We claim that the family $\{P_{g_\xi} : \xi < \alpha^\omega\}$ is equivalent to the usual $l^1 \alpha^\omega$ base. In order to prove this we remark first the following auxiliaries:

(a) For $n < \omega$ and $\xi < \alpha^\omega$, denoting $g^\xi(n) = g_\xi \mid S \setminus \bigcup_{i=1}^n U_i$ we have that

$$P_n \circ T^* \circ P(g^\xi(n)) = 0.$$

In fact, let $\mu \in L^1\{0,1\}^{\alpha_n}$. Then

$$\int P_n \circ T^* \circ P(g^\xi(n))d\mu = \int g^\xi(n)dP^* \circ T(\mu)$$

$$= \int g^\xi(n)d\mu_1 + \int g^\xi(n)d\mu_2$$

where $\mu_1 = P^* \circ T(\mu) | \bigcup_{i=1}^\omega U_i$ and $\mu_2 = P^* \circ T(\mu) | S \setminus \bigcup_{i=1}^\omega U_i$ and since $P^* \circ T(\mu)$ is absolutely continuous with respect to the measure λ_n and $g^\xi(n) | \overline{\bigcup_{i=1}^\omega U_i} = 0$ we have that

$$\int g^\xi(n)d\mu_2 = 0.$$

Also, from the regularity of μ_1 we have that

$$\int g^\xi(n)d\mu_1 = \lim_{k\to\infty} \int \sum_{j=n+1}^{\kappa} g_j^\xi d\mu_1 = \lim_{k\to\infty} \sum_{j=n+1}^{k} \int g_j^\xi d\mu_1$$

but

$$\int g_j^\xi d\mu_1 = \int g_j^\xi d(P^* \circ T(\mu)) = \int P_n \circ T^* \circ P(g_j^\xi)d\mu = 0.$$

The last equality holds from the inductive assumption (iii).

(b) If $x = \alpha_1\Pi_{M_1} + \cdots + \alpha_n\Pi_{M_n}$ is an element of $L^1\{-1,1\}^I$ and $J \subset I$, then if we set

$$y = \sum \{\alpha_i \Pi_{M_i} : M_i \not\subset J\},$$

$\|y\| \leqq 2\|x\|$ holds. In fact, if

$$E_J^1 : L^1\{-1,1\}^I \to L^1\{-1,1\}^J$$

is the conditional expectation, then $y = (I - E_J^1)(x)$ and so

$$\|y\| \leqq \|I - E_J^1\| \cdot \|x\| \leqq 2\|x\|.$$

In order to prove that the family $\{P_{g_\xi} : \xi < \alpha^\omega\}$ is equivalent to the usual base of $l^1\alpha^\omega$ it is enough to prove that the family $\{T^* \circ P(g_\xi) : \xi < \alpha^\omega\}$ has this property.

Let $\xi_1 < \xi_2 < \cdots < \xi_\kappa < \alpha^\omega$ and $r_1, r_2, \cdots, r_\kappa$ be given real numbers, and $\varepsilon > 0$. Then there is $n < \omega$ such that $\{g_n^{\xi_j} : j = 1, \cdots, \kappa\}$ are pairwise different and so there is a $x \in L^1\{-1,1\}^{\alpha_n}$ such that $\|x\| \leqq 1$, $x = \alpha_1\Pi_{M_1} + \cdots + \alpha_\lambda\Pi_{M_\lambda}$ and

$$\left| \int \sum_{j=1}^{\kappa} r_j E_n^\infty \circ T^* \circ P(g_n^{\xi_j})dx \right| \geqq \frac{\delta}{172} \sum_{j=1}^{\kappa} |r_j| - \varepsilon.$$

We set $y = \Sigma\{\alpha_i \Pi_{M_i} : M_i \not\subset I_n\}$. Then by inductive assumption (ii) we have that

$$\int P_n \circ T^* \circ P(g_n^{\xi_j})dx = \int P_n \circ T^* \circ P(g_n^{\xi_j})dy \qquad \text{for all } j = 1, 2, \cdots, \kappa.$$

Also, $g_{\xi_j} = \Sigma_{i=1}^{n} g_i^{\xi_j} + g^{\xi_j}(n)$, so using the previous (a) and (b) we have

$$\left\| \sum_{j=1}^{\kappa} r_j T^* \circ P(g_{\xi_j}) \right\| \geqq \frac{1}{2} \left| \sum_{j=1}^{\kappa} r_j \int P_n \circ T^* \circ P\left(\sum_{i=1}^{n-1} g_i^{\xi_j} \right) dy \right.$$

$$\left. + \sum_{j=1}^{\kappa} r_j \int P_n \circ T^* \circ P(g_n^{\xi_j})dy + \sum_{j=1}^{\kappa} r_j \int P_n \circ T^* \circ P(g^{\xi_j}(n)dy \right|$$

$$= \frac{1}{2} \left| \sum_{j=1}^{\kappa} r_j \int P_n \circ T^* \circ P(g_n^{\xi_j})dy \right| \geqq \frac{\delta}{354} \sum_{j=1}^{\kappa} |r_j| - \frac{\varepsilon}{2}$$

and the proof is complete.

We need, also, the following easy result of the cardinals arithmetic.

2.8. Lemma. *Let α be a cardinal such that $\alpha > 2^{\omega}$ and $\alpha^{\omega} > \alpha$. Then there is a cardinal β such that β is ω^+ inaccessible,* $\mathrm{cf}(\beta) = \omega$ *and $\beta^{\omega} = \alpha^{\omega}$.*

Proof. We set $\beta = \min\{\gamma : \gamma \leqq \alpha,\ \gamma^{\omega} \geqq \alpha\}$. Since $\alpha > 2^{\omega}$ it follows that $\beta > 2^{\omega}$ and β is ω^+ inaccessible, and we easily verify that $\mathrm{cf}(\beta) = \omega$.

2.9. Theorem. *Let X be an injective Banach space and $\{\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots : n < \omega\}$ be a sequence of cardinals such that $l^1\alpha_n$ is isomorphic to a subspace of X for all $n < \omega$. Then setting $\alpha = \sup\{\alpha_n : n < \omega\}$, X contains isomorphically a copy of the space $l^1\alpha^{\omega}$.*

Proof. If $\alpha \leqq 2^{\omega}$ then the result follows from Rosenthal's theorem, that $l^{\infty}\omega$ is isomorphic to a subspace of X[10].

Also, if $\alpha^{\omega} = \alpha$ then there is a n_0 such that $\alpha_{n_0} = \alpha$. So we assume that $\alpha > 2^{\omega}$ and $\alpha^{\omega} > \alpha$; and from Lemma 2.8 we can consider that α is ω^+ inaccessible cardinal and $\mathrm{cf}(\alpha) = \omega$. Let $\{\beta_1, \beta_2, \cdots, \beta_n, \cdots : n < \omega\}$ be a strictly increasing sequence of regular ω^+ inaccessible cardinals with $\sup_{n<\omega} \beta_n = \alpha$. Since $\beta_n < \alpha$ we have that $l^1\beta_n$ is isomorphic to a subspace of X. Furthermore, from [6], see also [7] theorem 2.e.3, it follows that $l^1\beta_n$ is $(1+\varepsilon)$-isomorphic to a subspace of X for all $\varepsilon > 0$. So from [8] there is an isomorphism

$$T_n : \left(\sum_{\xi < \alpha_n} \oplus L_{\xi}^1\{0,1\}^{\beta_n} \right)_1 \to X^*$$

with $\|T\| = 1$ and $\|T^{-1}\|^{-1} \geqq \frac{1}{2}$.

Let S be an extremally disconnected compact space such that X is a complemented subspace of $C(S)$ and P be a projection onto X. Let, also, $\{\mu_\gamma : \gamma \in \Gamma\}$ be a family of finite positive pairwise singular measures in $M^1(S)$ such that

$$M(S) = \left(\sum \oplus L^1(\mu_\gamma)\right)_1.$$

For the family of operators

$$P^* \circ T_n : \left(\sum_{\xi < \alpha_n} \oplus L^1_\xi\{0,1\}^{\beta_n}\right)_1 \to \left(\sum_{\gamma \in \Gamma} \oplus L^1(\mu_\gamma)\right)_1$$

we apply Lemma 2.3 and we find sequences $\{\xi_n : n < \omega\}$, $\{\Delta_n : n < \omega\}$ such that the conclusion of Lemma 2.3 be satisfied. So if we set $T = \Sigma_{n<\omega} \oplus T_n$, $\Delta = \bigcup_{n<\omega} \Delta_n$ then the operator

$$P_\Delta \circ P^* \circ T : \left(\sum \oplus L^1_{\xi_n}\{0,1\}^{\beta_n}\right)_1 \to \left(\sum_{\gamma \in \Delta} \oplus L^1(\mu_\gamma)\right)_1$$

is an isomorphism and for every $n < \omega$

$$P_\Delta \circ P^* \circ T(L^1_{\xi_n}\{0,1\}^{\beta_n}) \hookrightarrow \left(\sum_{\gamma \in \Delta_n} \oplus L^1(\mu_\gamma)\right)_1.$$

Let $K_n = \{\Pi_M : M \in \mathcal{P}_\omega(\beta_n)\}$ be the weakly discrete subset of $L^1\{0,1\}^{\beta_n}$ and since $K_n \cup \{0\}$ is weakly compact, it follows that there is a measure $\lambda_n \in (\Sigma_{\gamma \in \Delta_n} \oplus L^1(\mu_\gamma))_1$ such that the set

$$\{P_\Delta \circ P^* \circ T(\Pi_M) : M \in \mathcal{P}_\omega(\beta_n)\}$$

and the measure λ_n satisfies the conclusion of Lemma 2.1. Since the sequence $\{\lambda_n : n < \omega\}$ is equivalent to the usual base of $l^1\omega$ from Grothendick's theorem, there is a sequence $\{U_{n_1}, U_{n_2}, \cdots, U_{n_\kappa} : \kappa < \omega\}$ of pairwise disjoint clopen subsets of S and $\delta > 0$ such that $\lambda_{n_\kappa}(U_{n_\kappa}) > \delta$.

Applying now Lemma 2.1 for every $\kappa < \omega$ we find $K'_{n_\kappa} \subset K_{n_\kappa}$ with $|K'_{n_\kappa}| = \beta_n$ such that for every $\Pi_M \in K'_{n_\kappa}$

$$|P^* \circ T(\Pi_M)|(U_{n_\kappa}) > \delta.$$

So from Lemma 2.7 it follows that $l^1\beta^\omega$ is isomorphic to a subspace of X and the proof of the theorem is complete.

An immediate consequence of Theorems 2.5 and 2.8 it the following.

2.10. Corollary. *Let X be an injective subspaceof $L^\infty(\mu)$ for some finite measure μ with* $\dim X = \alpha$. *Then $l^1\alpha$ is isomorphic to a subspace of X.*

PROOF. If $\dim X = 2^\omega$ then the result follows from the isomorphic embedding of $l^\infty\omega$.

If $\dim X = \alpha > 2^\omega$ then either α is ω^+ inaccessible cardinal, or there is a cardinal β, ω^+ inaccessible with $\mathrm{cf}(\beta) = \omega$ and $\beta^\omega \geqq \alpha$. So applying Theorems 2.6 and 2.9 we get the desired result.

Also, from Theorem 2.8 follows the next

2.11. COROLLARY. *If X is an injective Banach space with* $\dim X = \alpha$ *and $l^1\alpha$ is isomorphic to a subspace of X then $\alpha^\omega = \alpha$.*

2.12. REMARK. From the last corollary it follows that in problem 7 of [9] (b) implies (a). In [1] we have also proved question (a) under the G.C.H. Finally in [2] we have proved the equivalence of questions (b) and (c).

2.13. REMARK. What we need for the proof of Theorem 2.8 is that the space X is a quotient space of a space $C(S)$ with S ω-complete space (i.e. there is a basis for the topology of S such that if $\{U_1, U_2, \cdots, U_n, \cdots : n < \omega\}$ are basic open sets, then $\overline{\bigcup_{n<\omega} U_n}$ is also open).

2.14. REMARK. In the case where X is isomorphic to the space $C(S)$ for S an extremally disconnected compact space, there is a complete affirmative answer in Rosenthal's problem. In fact, recently Balcar has proved that every such space S can be mapped continuously onto the space $\{0, 1\}^{w(S)}$ and so $l^1\alpha$ is isomorphic to a subspace of $C(S)$ where $\alpha = w(S)$.

For an arbitrary injective Banach space X with $\dim X = \alpha$ we know that $l^1\alpha$ is isomorphic to a subspace of X, if X is isomorphic to a conjugate Banach space Y^*. The proof of this result will appear elsewhere.

In the following theorem we summarize all the results which we know concerning the possibility of embedding $l^1\alpha$ into an injective Banach space X where $\dim X = \alpha$.

2.15. THEOREM. *Let X be an injective Banach space with* $\dim X = \alpha$. *If X has one of the following properties, then $l^1\alpha$ is isomorphic to a subspace of X.*

(a) *X is isomorphic to $C(S)$ for some S extremally disconnected compact space.*

(b) *Cardinals α and* $\mathrm{cf}(\alpha)$ *are $\Sigma(X)$ inaccessible.*

(c) *There is a bounded linear operator $T : X \to L^\infty(\mu)$ with μ finite measure and* $(\dim T(X))^\omega \geqq \alpha$.

(d) *There is a sequence $\{\beta_1, \beta_2, \cdots, \beta_n, \cdots : n < \omega\}$ of cardinals such that $l^1\beta_n$ is isomorphic to a subspace of X and $\Pi_{n<\omega}\beta_n \geqq \alpha$.*

(e) *X is isomorphic to a conjugate Banach space Y^*.*

(f) *There is a weakly compact K, subset of X^*, with* $(\text{dens.}\, K)^\omega \geqq \alpha$.

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